M464 - Introduction To Probability II - Homework 11

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April 10, 2014

Chapter 6

Exercises

1.3 A population of organisms evolves as follows. Each organism exists, independent of the other organisms, for an exponentially distributed length of time with parameter θ, and then splits into two new organisms, each of which exists, independent of the other organisms, for an exponentially distributed length of time with parameter θ, and then splits into two new organisms, and so on. Let $X(t)$ denote the number of organisms existing at time t. Show that $X(t)$ is a Yule process.

Solution: Let us check two conditions for a Yule process: (i) $X(0) = 1$, and (ii) all members of the population have the same birth rate β, i.e., the population birth rate is $\lambda_n = n\beta$.

(i) is obvious since we need at least one individual for the process to start.
(ii) Let us compute the probability of an individual reproducing for a small interval h. We know that $T \sim Exp(\theta)$ is the distribution of the time of reproduction. Then:

$$\begin{aligned}
Pr\{\text{an individual reproducing in an interval of length } h\} &= Pr\{T \leq h\} && \text{Distribution of reproduction time} \\
&= 1 - e^{-\theta h} && \text{Exponential cdf} \\
&= 1 - \sum_{k=0}^{\infty} \frac{(-\theta h)^k}{k!} && \text{Taylor expansion of } e^x \\
&= 1 - \left[1 - \theta h + \sum_{k=2}^{\infty} \frac{(-\theta h)^k}{k!}\right] && \text{taking first two terms} \\
&= \theta h + \sum_{k=2}^{\infty} \frac{(-\theta h)^k}{k!} \text{ algebra}
\end{aligned}$$

Now, the function $f(h) = \sum_{k=2}^{\infty} \frac{(-\theta h)^k}{k!}$ is little o of h since dividing $f(h)$ by h we get:

$$\frac{f(h)}{h} = \frac{\sum_{k=2}^{\infty} \frac{(-\theta h)^k}{k!}}{h} = \sum_{k=2}^{\infty} \frac{\frac{(-\theta h)^k}{k!}}{h} = \sum_{k=2}^{\infty} \frac{(-\theta h)^k}{h \cdot k!} = \sum_{k=2}^{\infty} \frac{(-\theta)^k h^{k-1}}{k!} \to 0 \text{ as } h \to 0$$

Since each term is multiplied by a positive power of h. Hence, the infinitesimal rate (i.e., dividing by h) of reproduction of each individual (as $h \to 0$) is θ. This means that the population birth rate is $\lambda_n = n\theta$, where n is the number of individuals.

Problems

1.1 Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over $[0, 1)$. Show that $Pr\{X(U) = k\} = \frac{p^k}{\beta k}$ for $k = 1, 2, \ldots$, with $p = 1 - e^{-\beta}$

Solution: Since $X(t)$ is a Yule process, $P_k(t) = e^{-\beta t}(1 - e^{-\beta t})^{k-1}, k \geq 1$. By the law of total probability:

$$Pr\{X(U) = k\} = \int_0^1 e^{-\beta t}(1 - e^{-\beta t})^{k-1} \cdot f_U(t)dt = \int_0^1 e^{-\beta t}(1 - e^{-\beta t})^{k-1} \cdot 1dt = \int_0^1 e^{-\beta t}(1 - e^{-\beta t})^{k-1}dt$$

Making the change of variables: $u = 1 - e^{-\beta t} \Rightarrow du = \beta e^{-\beta t}dt$, and the limits: if $t = 0$ then $u = 0$ and if $t = 1$ then $u = 1 - e^{-\beta}$ (here β is a positive number). Then:

$$Pr\{X(U) = k\} = \int_0^1 e^{-\beta t}(1 - e^{-\beta t})^{k-1}dt = \int_0^{1-e^{-\beta}} \frac{1}{\beta} u^{k-1} du = \frac{1}{\beta}\left[\frac{u^k}{k}\right]_0^{1-e^{-\beta}} = \frac{(1 - e^{-\beta})^k}{\beta k} = \frac{p^k}{\beta k}, \text{ letting } p = 1 - e^{-\beta}$$

1.3 Consider a population comprising a fixed number N of individuals. Suppose that at time $t = 0$ there is exactly one *infected* individual and $N - 1$ *susceptible* individuals in the population. Once infected, an individual remains in that state forever. In any short time interval of length h, *any given infected person* will transmit the disease to *any given susceptible person* with probability $\alpha h + o(h)$. (The parameter α is the *individual infection rate.*) Let $X(t)$ denote the number of infected individuals in the population at time $t \geq 0$. Then $X(t)$ is a pure birth process on the states $0, 1, \dots, N$. Specify the birth parameters.

Solution: Let λ_i be the birth parameters. Since this is a finite state, we need to specify a finite number of such birth parameters: $\lambda_0, \dots, \lambda_N$. Clearly, $\lambda_0 = \lambda_N = 0$, i.e., if no individual is infected then there can be no new infections and if all individuals are infected, then no further infections occur. Following a similar reasoning to that of section 1.3, we can compute the birth parameters by noting that (for $0 \leq k < N$):

$$Pr\{X(t+h) - X(t) = 1 | X(t) = k\} = \binom{N-k}{1}\binom{k}{1} [\alpha h + o(h)] [1 - \alpha h + o(h)]^{k(N-k)-1} = (N-k)k [\alpha h + o(h)] [1 - \alpha h + o(h)]^{k(N-k)-1}$$

That is, the chance of getting exactly one new infection given that there are k individuals already infected, corresponds to the probability of exactly one pair of infected and susceptible individuals ($(N-k)k$) resulting in a new infected individual ($[\alpha h + o(h)]$) while all other possible pairs results in no new infections ($[1 - \alpha h + o(h)]^{k(N-k)-1}$).
By the Binomial Theorem:

$$Pr\{X(t+h) - X(t) = 1 | X(t) = k\} = (N-k)k [\alpha h + o(h)] [1 - \alpha h + o(h)]^{k(N-k)-1} = (N-k)k [\alpha h + o(h)]$$

Now we can find the birth parameters by letting h tend to zero. For N we already know that $\lambda_N = 0$. For $0 \leq k < N$:

$$\lambda_k = \lim_{h \to 0} \frac{Pr\{X(t+h) - X(t) = 1 | X(t) = k\}}{h} = \lim_{h \to 0} \frac{(N-k)k [\alpha h + o(h)]}{h} = \lim_{h \to 0} (N-k)k\alpha + \frac{(N-k)k o(h)}{h} = (N-k)k\alpha$$

Thus, the birth parameters are given by $\boxed{\lambda_k = (N-k)k\alpha}$. Interestingly enough, this expression works for all values of k such that $0 \leq k \leq N$, further confirming facts already deduced such as $\lambda_N = (N-N)N\alpha = 0$ and $\lambda_0 = (N-0)0\alpha = 0$.